

THE POSITIVE CONTRACTIVE PART OF A NONCOMMUTATIVE L^p -SPACE IS A COMPLETE JORDAN INVARIANT

CHI-WAI LEUNG, CHI-KEUNG NG, AND NGAI-CHING WONG

ABSTRACT. Let $1 \leq p \leq +\infty$. We show that the positive part of the closed unit ball of a non-commutative L^p -space, as a metric space, is a complete Jordan $*$ -invariant for the underlying von Neumann algebra.

1. INTRODUCTION

Given a von Neumann algebra M , celebrated results of R. V. Kadison showed that several partial structures of M can recover the von Neumann algebra up to Jordan $*$ -isomorphisms. In particular, each of the following is a complete Jordan $*$ -invariant of M : the Banach space structure of the self-adjoint part M_{sa} of M ([5, Theorem 2]), the ordered vector space structure of M_{sa} ([5, Corollary 5]) and the topological convex set structure of the normal state space of M ([6, Theorem 4.5]).

Let $p \in [1, +\infty]$, and let $L^p(M)$ be the non-commutative L^p -space associated to M with the canonical cone $L^p(M)_+$. If M is semi-finite, P.-K. Tam showed in [15] that the ordered Banach space $(L^p(M)_{\text{sa}}, L^p(M)_+)$ characterises M up to Jordan $*$ -isomorphisms. In the case when M is σ -finite (but not necessarily semi-finite) and $p = 2$, the corresponding result follows from a result of A. Connes (namely, [3, Théorème 3.3]). On the other hand, extending results of B. Russo ([12]) and F. J. Yeadon ([16]), D. Sherman showed in [13] that the Banach space $L^p(M)$ is also a complete Jordan $*$ -invariant for a general von Neumann algebra M when $p \neq 2$.

Along this line, we show in this article that the underlying metric space structure of the positive contractive part

$$L^p(M)_+^1 := L^p(M)_+ \cap L^p(M)^1 \quad (1 \leq p \leq +\infty)$$

of $L^p(M)$ is also a complete Jordan $*$ -invariant of M , where $L^p(M)^1$ is the closed unit ball. More precisely, we will show in Theorem 3.11 that two arbitrary von Neumann algebras M and N are Jordan $*$ -isomorphic whenever there exist a bijective isometry Φ from $L^p(M)_+^1$ onto $L^p(N)_+^1$, i.e.,

$$\|\Phi(x) - \Phi(y)\| = \|x - y\| \quad (x, y \in L^p(M)_+^1).$$

Notice that the closed unit ball $L^2(M)^1$ itself is not a complete Jordan $*$ -invariant (since for any infinite dimensional von Neumann algebra M with a separable predual, one has $L^2(M) \cong \ell^2$), but its positive part is a Jordan $*$ -invariant.

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2. PRELIMINARIES

Throughout this article, if E is a subset of a normed space X and $\lambda > 0$, we set

$$E^\lambda := \{x \in E : \|x\| \leq \lambda\}.$$

In the following, we will briefly recall (mainly from [11]) notations concerning non-commutative L^p -spaces. Let M be a (complex) von Neumann algebra on a (complex) Hilbert space \mathfrak{H} and $\alpha : \mathbb{R} \rightarrow \text{Aut}(M)$ be the modular automorphism group. Then the von Neumann algebra crossed product $\check{M} := M \bar{\rtimes}_\alpha \mathbb{R}$ is semi-finite and we fix a normal faithful semi-finite trace τ on \check{M} . The *measure topology* on \check{M} (as introduced by E. Nelson in [9]) is given by a neighborhood basis at 0 of the form

$$U(\epsilon, \delta) := \{x \in \check{M} : \|xp\| \leq \epsilon \text{ and } \tau(1 - p) \leq \delta, \text{ for a projection } p \in \check{M}\}.$$

The completion, $L_0(\check{M}, \tau)$, of \check{M} with respect to this topology is a $*$ -algebra extending the $*$ -algebra structure on \check{M} .

One may identify $L_0(\check{M}, \tau)$ with a collection of closed and densely defined operators on $L^2(\mathbb{R}; \mathfrak{H})$ affiliated with \check{M} . More precisely, suppose that T is such a closed operator on $L^2(\mathbb{R}; \mathfrak{H})$ and that $|T|$ is the absolute value of T with the spectral measure $E_{|T|}$. Then T corresponds (uniquely) to an element in $L_0(\check{M}, \tau)$ if and only if $\tau(1 - E_{|T|}([0, \lambda])) < +\infty$ when λ is large enough. In this case, the $*$ -operation on $L_0(\check{M}, \tau)$ coincides with the adjoint. Moreover, the addition and the multiplication on $L_0(\check{M}, \tau)$ are the closures of the corresponding operations for densely defined closed operators. We denote by $L_0(\check{M}, \tau)_+$ the set of all positive self-adjoint (but not necessarily bounded) operators in $L_0(\check{M}, \tau)$.

The dual action $\hat{\alpha} : \mathbb{R} \rightarrow \text{Aut}(\check{M})$ of α extends to an action on $L_0(\check{M}, \tau)$ by $*$ -automorphisms. For any $p \in [1, +\infty]$, we set, as in the literature,

$$L^p(M) := \{T \in L_0(\check{M}, \tau) : \hat{\alpha}_s(T) = e^{-s/p}T, \text{ for all } s \in \mathbb{R}\}.$$

Denote by $L^p(M)_{\text{sa}}$ the set of all self-adjoint operators in $L^p(M)$ and put

$$L^p(M)_+ := L^p(M) \cap L_0(\check{M}, \tau)_+.$$

If $T \in L_0(\check{M}, \tau)$ and $T = u|T|$ is the polar decomposition, then $T \in L^p(M)$ if and only if $u \in M$ and $|T| \in L^p(M)$.

In the case when $p \in (1, +\infty)$, the map that sends $x \in \check{M}_+$ to x^p extends to a map

$$\Lambda_p : L_0(\check{M}, \tau)_+ \rightarrow L_0(\check{M}, \tau)_+.$$

For any $T \in L_0(\check{M}, \tau)_+$, one has $T \in L^p(M)$ if and only if $\Lambda_p(T) \in L^1(M)$. There is a canonical identification of M_* with $L^1(M)$ that sends the positive part $M_{*,+}$ of M_* onto $L^1(M)_+$, and this induces a Banach space norm $\|\cdot\|_1$ on $L^1(M)$. The function defined by

$$(2.1) \quad \|T\|_p := \|\Lambda_p(|T|)\|_1^{1/p}$$

is a norm on $L^p(M)$ that turns it into a Banach space. On the other hand, one may identify M with $L^\infty(M)$ (as ordered Banach spaces) through the canonical inclusion $M \subseteq \check{M} \subseteq L_0(\check{M}, \tau)$.

3. RESULTS AND QUESTIONS

3.1. The case of $p = +\infty$.

Proposition 3.1. *If $\Phi : M_+^1 \rightarrow N_+^1$ is a bijective isometry, then $\Psi : x \mapsto \Phi(x + \frac{1}{2}) - \frac{1}{2}$ extends to a linear isometry from M_{sa} onto N_{sa} .*

We may then conclude from [5, Theorem 2] that $x \mapsto \Psi(1)\Psi(x)$ is a Jordan $*$ -isomorphism. In order to establish this proposition, we need the following stronger version of the Mazur-Ulam theorem, which was first proved in [8, Theorem 2] (see also [1, Theorem 14.1]).

Lemma 3.2. *Let U be a non-empty open connected subset of a normed space X and W be an open subset of a normed space Y . Then every isometry from U onto W can be extended uniquely to an affine isometry from X onto Y .*

Proof of Proposition 3.1. Let us first note that for any $x \in M_{sa}$, one has $x \in M_+^1$ if and only if $\|x - \frac{1}{2}\| \leq \frac{1}{2}$ (by considering the C^* -subalgebra generated by x and 1). Thus, $x \mapsto x - \frac{1}{2}$ is a bijective isometry from M_+^1 onto $M_{sa}^{\frac{1}{2}}$ and the map Ψ in the statement is a bijective isometry from $M_{sa}^{\frac{1}{2}}$ onto $N_{sa}^{\frac{1}{2}}$.

If $x \in M_{sa}^{\frac{1}{2}}$, then $\|x\| = \frac{1}{2}$ if and only if there exists $x' \in M_{sa}^{\frac{1}{2}}$ with $\|x - x'\| = 1$. This implies

$$\Psi(\{x \in M_{sa} : \|x\| = 1/2\}) = \{y \in N_{sa} : \|y\| = 1/2\}.$$

Consequently, $\Psi(0) = 0$ and Ψ will send the interior, $B_M(0, \frac{1}{2})$, of $M_{sa}^{\frac{1}{2}}$ onto the interior of $N_{sa}^{\frac{1}{2}}$. By Lemma 3.2, $\Psi|_{B_M(0, \frac{1}{2})}$ extends to a linear isometry $\tilde{\Psi}$ from M_{sa} onto N_{sa} and the continuity of Ψ tells us that $\tilde{\Psi}|_{M_{sa}^{\frac{1}{2}}} = \Psi$. \square

Example 3.3. Let $M = \mathbb{C} \oplus_{\infty} \mathbb{C}$. The set M_+^1 equals the square in $\mathbb{R} \oplus_{\infty} \mathbb{R}$ with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(1, 0)$. If $\Phi_0 : \mathbb{R} \oplus_{\infty} \mathbb{R} \rightarrow \mathbb{R} \oplus_{\infty} \mathbb{R}$ is the clockwise rotation by 90 degree about the center $(\frac{1}{2}, \frac{1}{2})$, then the restriction Φ of Φ_0 on M_+^1 is a bijective isometry onto M_+^1 that sends $(0, 0)$ to $(0, 1)$. Hence, Φ itself cannot be extended to a linear map. However, if Ψ is as defined in Proposition 3.1, then $\Psi(1, 1) = \Phi(\frac{3}{2}, \frac{3}{2}) - (\frac{1}{2}, \frac{1}{2}) = (1, -1)$ and the map

$$(x, y) \mapsto \Psi(1, 1)\Psi(x, y) = (1, -1)(\Phi_0(x + 1/2, y + 1/2) - (1/2, 1/2)) = (y, x)$$

is a $*$ -automorphism of M .

3.2. The case of $p = 1$.

Proposition 3.4. *If there exists a bijective isometry Φ from $M_{*,+}^1$ onto $N_{*,+}^1$, then M and N are Jordan $*$ -isomorphic.*

Note, first of all, that one cannot use Lemma 3.2 for this case, since the interior of $M_{*,+}^1$ could be an empty set, e.g. when $M = L^{\infty}([0, 1])$.

For any $\mu \in M_{*,+}$, we denote by $\text{supp } \mu$ the support projection of μ in M . Recall that for any $\mu, \nu \in M_{*,+}$, we have

$$(3.1) \quad \|\mu - \nu\| = \|\mu\| + \|\nu\| \quad \text{if and only if} \quad \text{supp } \mu \cdot \text{supp } \nu = 0.$$

In order to obtain Proposition 3.4, we need the following lemma.

Lemma 3.5. *If N contains three non-zero orthogonal projections q_1, q_2 and q_3 , then the bijective isometry Φ in Proposition 3.4 will send 0 to 0.*

Proof. Suppose on the contrary that $\Phi(0) \neq 0$. Let us first show that $\text{supp } \Phi(0) = 1$. Indeed, if it is not the case, one can find $\mu \in M_{*,+}^1$ such that $\|\Phi(\mu)\| = 1$ and $\text{supp } \Phi(\mu) \leq 1 - \text{supp } \Phi(0)$, which, together with (3.1), gives the contradiction that

$$1 \geq \|\mu - 0\| = \|\Phi(\mu) - \Phi(0)\| = \|\Phi(\mu)\| + \|\Phi(0)\| > 1.$$

As a result, $\Phi(0)(q_k) > 0$ for $k = 1, 2, 3$. We may also assume, without loss of generality, that $\Phi(0)(q_1) \leq \|\Phi(0)\|/3$ because

$$\sum_{k=1}^3 \Phi(0)(q_k) \leq \|\Phi(0)\|.$$

Now, pick any $\nu \in M_{*,+}^1$ with $\|\Phi(\nu)\| = 1$ and $\text{supp } \Phi(\nu) \leq q_1$. Since $1 - 2q_1$ is a unitary and $\|\Phi(\nu) - \Phi(0)\| = \|\nu\| \leq 1$, one arrives at the following contradiction:

$$1 \geq |(\Phi(\nu) - \Phi(0))(q_1 - (1 - q_1))| = |1 - \Phi(0)(q_1) + \Phi(0)(1 - q_1)| = 1 + \|\Phi(0)\| - 2\Phi(0)(q_1) > 1.$$

□

Consequently, if N contains three non-zero orthogonal projections, then Φ induces an isometric bijection from the normal state space of M to that of N , and hence, we may conclude that M and N are Jordan $*$ -isomorphic by using [7, Theorem 3.4]. For the benefit of the readers, we will instead go through briefly the argument of [7, Theorem 3.4] by recalling the following two lemmas. These two lemmas are also needed in the case of $p \in (0, +\infty)$ below.

Let us recall that a bijection Γ from the lattice of projections in M to that of N is called an *orthoisomorphism* if for any projections p and q in M , one has

$$pq = 0 \quad \text{if and only if} \quad \Gamma(p)\Gamma(q) = 0.$$

Lemma 3.6. ([7, Lemma 3.1(a)]) *Suppose that Ψ is a bijection from the normal state space of M to that of N , which is biorthogonality preserving in the sense that for any normal states μ and ν of M , one has*

$$\text{supp } \mu \cdot \text{supp } \nu = 0 \quad \text{if and only if} \quad \text{supp } \Psi(\mu) \cdot \text{supp } \Psi(\nu) = 0.$$

Then there is an orthoisomorphism $\tilde{\Psi}$ from the lattice of projections in M to that of N satisfying $\tilde{\Psi}(\text{supp } \mu) = \text{supp } \Psi(\mu)$ for any normal state μ on M .

A second lemma that we need is the following possibly well-known variant of a theorem of H. A. Dye in [4] (see e.g. [7, Lemma 2.2(a)]). Note that an assumption of not having type I_2 summand is needed for the original version of Dye's theorem. However, the variant here has a weaker conclusion and does not need the assumption concerning the absence of type I_2 summand.

Lemma 3.7. *If there exists an orthoisomorphism from the lattice of projections in M to that of N , then M and N are Jordan $*$ -isomorphic.*

Proof of Proposition 3.4. Let us first consider the case when N contains three non-zero orthogonal projections. Then by Lemma 3.5, the map Φ restricts to an isometric bijection Ψ from

the normal state space of M to that of N . Moreover, (3.1) implies that Ψ is biorthogonality preserving. Now, the conclusion follows from Lemmas 3.6 and 3.7.

In the case when M contains three non-zero orthogonal projections, one obtains the same conclusion by considering the bijective isometry Φ^{-1} .

Suppose that neither M nor N contains three non-zero orthogonal projections. Then M and N can only be \mathbb{C} , $\mathbb{C} \oplus_{\infty} \mathbb{C}$ or $M_2(\mathbb{C})$. Observe that the Hausdorff dimensions of the quasi-state space of \mathbb{C} , $\mathbb{C} \oplus_{\infty} \mathbb{C}$ and $M_2(\mathbb{C})$ are 1, 2 and 4 respectively. Since a bijective isometry preserves Hausdorff dimensions, we conclude that M and N are $*$ -isomorphic. \square

3.3. A preparation for the case of $p \in (1, +\infty)$.

Proposition 3.8. *Let $p \in (1, +\infty)$. Suppose that M and N are two von Neumann algebras such that either $M \neq \mathbb{C}$ or $N \neq \mathbb{C}$. Then any bijective isometry $\Phi : L^p(M)_+^1 \rightarrow L^p(N)_+^1$ extends to a linear isometry from $L^p(M)_{\text{sa}}$ onto $L^p(N)_{\text{sa}}$.*

Notice that $L^p(M)_{\text{sa}}$ and $L^p(N)_{\text{sa}}$ are strictly convex Banach spaces for $p \in (1, +\infty)$ (see e.g., Section 5 of [10]). We recall the following well-known fact concerning strictly convex Banach spaces.

Lemma 3.9. *Let X_1 and X_2 be Banach spaces such that X_2 is strictly convex. Then every isometry from a convex subset K of X_1 into X_2 is automatically an affine map.*

In fact, we only need to verify that $f((x+y)/2) = (f(x) + f(y))/2$, for any $x \neq y$ in K . By “shifting” K and f , one may assume that $y = 0$ and $f(0) = 0$. Under this assumption, we have $\|f(z)\| = \|z\|$ ($z \in K$) and

$$(3.2) \quad \|f(x) - f(x/2)\| = \|x - x/2\| = \|f(x)\|/2 = \|f(x)\| - \|x\|/2 = \|f(x)\| - \|f(x/2)\|.$$

The strict convexity of X_2 gives $f(x) - f(x/2) \in \mathbb{R} \cdot f(x/2)$. This, together with the last two equalities in (3.2), will produce $f(x) = 2f(x/2)$.

The following lemma is an analogue of [7, Proposition 3.7]. Note that we consider in this lemma bijective isometries between the contractive parts instead of those between the norm-one parts of K_1 and K_2 as in [7]. Moreover, we have a more general setting here.

Lemma 3.10. *Let X_1 and X_2 be strictly convex real Banach spaces of dimensions at least two (could be infinite). Let $\mathcal{F} : \mathbb{R}_+^{(4)} \rightarrow \mathbb{R}_+$ be a function satisfying*

$$\mathcal{F}(t, t, 0, t) = 0 \quad (t \in \mathbb{R}_+).$$

For $k = 1, 2$, suppose that $K_k \subseteq X_k$ is a closed and proper cone in X_k which is \mathcal{F} -generating, in the sense that for any $x \in X_k$, there exist unique elements $x_+, x_- \in K_k$ with

$$x = x_+ - x_- \quad \text{and} \quad \mathcal{F}(\|x\|, \|x_+\|, \|x_-\|, \|x_+ + x_-\|) = 0.$$

Then there are canonical bijective correspondences amongst the following (given by restrictions):

- the set \mathcal{I} of real linear isometries from X_1 onto X_2 that send K_1 onto K_2 .
- the set \mathcal{I}_B of bijective isometries from K_1^1 onto K_2^1 .
- the set \mathcal{I}_K of bijective isometries from K_1 onto K_2 .

Proof. If $\Phi \in \mathcal{J}$, then obviously $\Phi|_{K_1^1} \in \mathcal{J}_B$. The assignment $\Phi \mapsto \Phi|_{K_1^1}$ is an injection from \mathcal{J} to \mathcal{J}_B because K_1^1 linearly spans X_1 .

Suppose that $\Psi \in \mathcal{J}_B$. We put

$$S_i := \{u \in K_i : \|u\| = 1\} \quad (i = 1, 2).$$

The set of extreme points of K_i^1 is $S_i \cup \{0\}$ (since X_i is strictly convex). By Lemma 3.9, the map Ψ is affine and hence $\Psi(0) \in S_2 \cup \{0\}$. If $\Psi(0) \in S_2$, then there is a sequence $\{v_i\}_{i \in \mathbb{N}}$ in $S_2 \setminus \{\Psi(0)\}$ with $\|v_i - \Psi(0)\| \rightarrow 0$ (as $\dim X_2 > 1$), and hence $\{\Psi^{-1}(v_i)\}_{i \in \mathbb{N}}$ is a sequence in S_1 norm-converging to 0, which is absurd. Thus, we know that $\Psi(0) = 0$. Define $\hat{\Psi} : K_1 \rightarrow K_2$ by

$$(3.3) \quad \hat{\Psi}(0) := 0 \quad \text{and} \quad \hat{\Psi}(u) := \|u\| \Psi(u/\|u\|) \quad (u \in K_1 \setminus \{0\}).$$

As Ψ is an affine map sending 0 to 0, we see that $\hat{\Psi}$ extends Ψ and that $\hat{\Psi}(tu) = t\hat{\Psi}(u)$ ($u \in K_1, t \in \mathbb{R}_+$). For any $u, v \in K_1$, if $\lambda := \|u\| + \|v\| + 1$, then

$$\|\hat{\Psi}(u) - \hat{\Psi}(v)\| = \left\| \lambda \Psi\left(\frac{u}{\lambda}\right) - \lambda \Psi\left(\frac{v}{\lambda}\right) \right\| = \lambda \left\| \frac{u}{\lambda} - \frac{v}{\lambda} \right\| = \|u - v\|.$$

Consequently, $\hat{\Psi} \in \mathcal{J}_K$. The assignment $\Psi \mapsto \hat{\Psi}$ is clearly injective.

Suppose that $\varphi \in \mathcal{J}_K$. Again, Lemma 3.9 implies that φ is affine, and will send extreme points of K_1 to extreme points of K_2 . However, as K_i is a proper cone, the only extreme point in K_i is zero ($i = 1, 2$) and we have $\varphi(0) = 0$. This means that φ is additive and positively homogeneous on K_1 . Hence,

$$(3.4) \quad \|\varphi(u) + \varphi(v)\| = \|\varphi(u + v)\| = \|u + v\| \quad (u, v \in K_1).$$

Let us define $\tilde{\varphi} : X_1 \rightarrow X_2$ by

$$\tilde{\varphi}(x) := \varphi(x_+) - \varphi(x_-) \quad (x \in X_1).$$

Since $\mathcal{F}(t, t, 0, t) = 0$ for all $t \in \mathbb{R}_+$, one has $x_+ = x$ and $x_- = 0$ whenever $x \in K_1$. Hence, $\tilde{\varphi}$ extends φ . Moreover, Relation (3.4) as well as $\|\tilde{\varphi}(x)\| = \|x\|$ ($x \in X_1$) implies

$$\mathcal{F}(\|\tilde{\varphi}(x)\|, \|\varphi(x_+)\|, \|\varphi(x_-)\|, \|\varphi(x_+) + \varphi(x_-)\|) = \mathcal{F}(\|x\|, \|x_+\|, \|x_-\|, \|x_+ + x_-\|) = 0,$$

and the uniqueness of $\tilde{\varphi}(x)_\pm$ ensures that $\tilde{\varphi}(x)_\pm = \varphi(x_\pm)$ ($x \in X_1$). Furthermore, for any $x, y \in X_1$, one has

$$\begin{aligned} \|\tilde{\varphi}(x) - \tilde{\varphi}(y)\| &= \|\varphi(x_+) - \varphi(x_-) - \varphi(y_+) + \varphi(y_-)\| = \|\varphi(x_+ + y_-) - \varphi(x_- + y_+)\| \\ &= \|(x_+ + y_-) - (x_- + y_+)\| = \|x - y\|. \end{aligned}$$

Applying the same arguments to

$$\psi := \varphi^{-1},$$

we will obtain a map $\tilde{\psi}$ from X_2 into X_1 satisfying $\tilde{\psi}(z)_\pm = \psi(z_\pm)$ ($z \in X_2$). For any z in X_2 , if we set $x := \tilde{\psi}(z) \in X_1$, then

$$\tilde{\varphi}(x) = \varphi(x_+) - \varphi(x_-) = \varphi(\psi(z_+)) - \varphi(\psi(z_-)) = z.$$

This ensures the surjectivity of $\tilde{\varphi}$. Hence, $\tilde{\varphi}$ is a bijective isometry sending 0 to 0, and the Mazur-Ulam theorem tells us that $\tilde{\varphi} \in \mathcal{J}$. It is easy to see that the canonical extension of $\tilde{\varphi}|_{K_1^1}$ to K_1 as in (3.3) coincides with φ . This completes the proof. \square

For any $T \in L^p(M)_{\text{sa}}$, we denote by $\text{supp } T$ the support projection of T , i.e. $\text{supp } T$ is the smallest projection p in M satisfying $T \cdot p = T$ (or equivalently, $p \cdot T = T$). Let us recall the following statements concerning $S, T \in L^p(M)_+$ from Fact 1.2 and Fact 1.3 of [11]:

- S1). $\text{supp } \Lambda_p(T) = \text{supp } T$;
 S2). $S \cdot T = 0$ if and only if $\text{supp } S \cdot \text{supp } T = 0$;
 S3). if $\text{supp } S \cdot \text{supp } T = 0$, then $\|S + T\|_p^p = \|S - T\|_p^p = \|S\|_p^p + \|T\|_p^p$;
 S4). if $p \neq 2$ and $\|S + T\|_p^p = \|S - T\|_p^p = \|S\|_p^p + \|T\|_p^p$, then $\text{supp } S \cdot \text{supp } T = 0$.

Proof of Proposition 3.8. We note, first of all, that if $M = \mathbb{C}$, then there are only two extreme points of $L^p(M)_+^1$ and hence the strict convexity of $L^p(N)_{\text{sa}}$ implies that there are only two extreme points of $L^p(N)_+^1$ (see the argument of Lemma 3.10), which gives $N = \mathbb{C}$. Therefore, the hypothesis actually implies that the dimensions of both M_{sa} and N_{sa} are at least two.

Let us first consider the case when $p \neq 2$, and define a map $\mathcal{F}_p : \mathbb{R}_+^{(4)} \rightarrow \mathbb{R}_+$ by

$$\mathcal{F}_p(a, b, c, d) := |a^p - b^p - c^p| + |d^p - b^p - c^p|.$$

For any $T \in L^p(M)_{\text{sa}}$, we know that $|T| \in L^p(M)_+$. We denote by T_+ and T_- the positive part and the negative part of the self-adjoint operator T respectively. As a closed operator, T_{\pm} is the closure of $\frac{|T| \pm T}{2}$. Hence, $T_{\pm} = \frac{|T| \pm T}{2}$ as elements in $L_0(\check{M}, \tau)$. This means that $T_{\pm} \in L^p(M)_+$ and satisfies $T_+ T_- = T_- T_+ = 0$. Now, we know from (S2) and (S3) that

$$\mathcal{F}_p(\|T\|, \|T_+\|, \|T_-\|, \|T_+ + T_-\|) = 0.$$

Conversely, suppose that $T \in L^p(M)_{\text{sa}}$ and $R, S \in L^p(M)_+$ such that $T = R - S$ and

$$\mathcal{F}_p(\|T\|, \|R\|, \|S\|, \|R + S\|) = 0.$$

Then by (S2) and (S4), we have $RS = 0$. Therefore, $(R + S)^2 = (R - S)^2 = T^2$, which implies that $R + S = |T|$ (because $R + S$ is a positive self-adjoint operator; see e.g. [2, Theorem 12]). Consequently,

$$R = T_+ \quad \text{and} \quad S = T_-.$$

This means that $L^p(M)_+$ is a \mathcal{F}_p -generating cone of $L^p(M)_{\text{sa}}$ and we may apply Lemma 3.10 to extend Φ to a real linear isometry from $L^p(M)_{\text{sa}}$ onto $L^p(N)_{\text{sa}}$.

For $p = 2$, we know from the proof of Lemma 3.10 that $\Phi(0) = 0$ and hence Φ restricts to a bijective isometry from the set of norm-one elements in $L^2(M)_+$ onto that of $L^2(N)_+$. Now, the conclusion follows from [7, Proposition 3.7]. \square

3.4. The proof for the case $p \in (1, +\infty)$ and the presentation of the main result.

Theorem 3.11. *Let M and N be two von Neumann algebras and let $p \in [1, +\infty]$. If there is a bijective isometry $\Phi : L^p(M)_+^1 \rightarrow L^p(N)_+^1$, then M and N are Jordan $*$ -isomorphic.*

Proof. The cases of $p = +\infty$ and $p = 1$ are proved in Proposition 3.1 (together with [5, Theorem 2]) and Proposition 3.4, respectively (through the canonical identifications of $L^1(M)$ and $L^\infty(M)$ with M_* and M). Moreover, the case of $p = 2$ is already established in [7, Corollary 3.11] (due to Proposition 3.8 and [7, Proposition 3.7]).

Now, we consider $p \in (1, +\infty) \setminus \{2\}$. Without loss of generality, we assume that either $M \neq \mathbb{C}$ or $N \neq \mathbb{C}$. By Proposition 3.8, Φ is an affine map with $\Phi(0) = 0$. Furthermore, it follows from Relation (2.1) that Λ_p induces a bijection from $L^p(M)_+^1$ onto $L^1(M)_+^1$ that sends the norm one part of $L^p(M)_+$ onto the norm one part, $\mathfrak{S}(M)$, of $L^1(M)_+$. Hence, Φ induces a bijection

$\hat{\Phi} : \mathfrak{S}(M) \rightarrow \mathfrak{S}(N)$ with $\hat{\Phi}(A) = \Lambda_p(\Phi(\Lambda_p^{-1}(A)))$. For any $A, B \in \mathfrak{S}(M)$, it follows from (S1), (S3) and (S4) that

$$\text{supp } A \cdot \text{supp } B = 0 \text{ if and only if } \left\| \frac{\Lambda_p^{-1}(A)}{2} + \frac{\Lambda_p^{-1}(B)}{2} \right\|_p^p = \left\| \frac{\Lambda_p^{-1}(A)}{2} - \frac{\Lambda_p^{-1}(B)}{2} \right\|_p^p = 2^{1-p}.$$

As Φ is both isometric and affine (as well as $\Phi(0) = 0$), we know that $\hat{\Phi}$ is a biorthogonality preserving bijection between the normal state spaces of M and N (through the identification $L^1(M) = M_*$). The conclusion now follows from Lemmas 3.6 and 3.7. \square

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(Chi-Wai Leung) DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, HONG KONG.

E-mail address: cwleung@math.cuhk.edu.hk

(Chi-Keung Ng) CHERN INSTITUTE OF MATHEMATICS AND LPMC, NANKAI UNIVERSITY, TIANJIN 300071, CHINA.

E-mail address: ckng@nankai.edu.cn

(Ngai-Ching Wong) DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL SUN YAT-SEN UNIVERSITY, KAOHSIUNG, 80424, TAIWAN.

E-mail address: `wong@math.nsysu.edu.tw`